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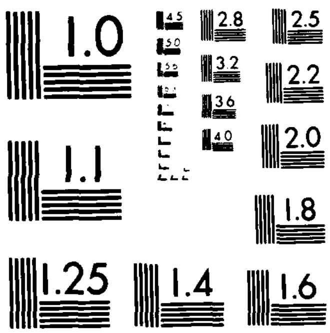

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EXPLICIT, NEARLY OPTIMAL, LINEAR RATIONAL APPROXIMATION WITH PREASSIGNED POLES +

by

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1. INTRODUCTION AND SUMMARY

In this paper we attempt to give a constructive, affirmative answer to each of the following questions.

1. Given a function f and an interval I, is it possible to tell a priori whether or not one can accurately approximate f via a low degree rational function?

2. Can such a rational function be easily constructed explicitly, so that one encounters no poles on the interval of approximation?

3. Can one use the Thiele algorithm to construct or evaluate this rational function?

4. Can one tel_ a priori, when we can expect the Thiele algorithm, the ε-algorithm, or the Padé method to produce an accurate low degree rational approximation?

5. Does the error of this rational function compare favorably with the error of the best possible rational approximation of the same degree?

Although we cannot give an affirmative answer to the above questions in all cases, we shall describe classes of analytic functions which house nearly all of the cases encountered by the author in applications, and for which the answer to each of the above questions is "Yes".

We shall develop a class of rational approximations for interpolation over $[-1,1]$, $[0,\infty]$ and $[-\infty,\infty]$. These rational approximations share many of the features of the sinc methods summarized in [17]. The interpolation points of these rationals are the same as the sinc points and the classes of functions which low degree rationals approximate accurately are the same as the classes which the sinc functions approximate accurately. Indeed the error bounds for e.g. approximation on $[-1,1]$ of functions analytic on the unit disc are the same as the sinc bounds, i.e. rationals have the same optimality properties as sinc methods. In using rationals instead of sinc functions, we lose many of the simple relations that sinc functions satisfy, such as orthogonality and ease of getting other formulas such as quadrature, approximation of derivatives, methods of solving differential equations, etc. However, the well-known rational function algorithms of Thiele [21] (the ρ-algorithm), Padé [12], and Shanks [15], Wynn [23] (the ϵ-algorithm) all share simple methods of prediction, which the sinc functions do not appear to possess. This paper provides an understanding in that it enables us to tell a-priori, when we can expect these algorithms to work effectively.

The spaces of functions for which the rationals provide accurate approximations are described precisely in Sec. 2 of the paper. One such space, consists, roughly of functions analytic on an interval with possible singularities at end-points of an interval, such that the functions are of class Lip_α ($\alpha > 0$) on the interval, i.e., functions which one encounters in nearly all cases in applications.

The rational approximations of this paper have the following additional features.

(a) There are no poles on the interval of approximation.

(b) The rational functions are linear in f, the function that is being approximated.

(c) They are nearly optimal. More precisely, we prove the following result:

THEOREM 1.1: Let $1 < p < \infty$, let $p' = p/(p-1)$, let U denote the unit disc in the complex plane, let g be in the Hardy space $H_p(U)$, and let $f(z) = (1 - z^2)g(z)$. Let $\mathbb{P}_n$ denote the space of polynomials of degree n and set

$$(1.1) \qquad \delta_N = \inf_{\mu \in \mathbb{P}_{2N+2}\,,\ \sigma \in \mathbb{P}_{2N+1}} \ \sup_{g \in H_p(U)\,,\ \|g\|_p = 1} \ \sup_{-1<x<1} \left| f(x) - \frac{\mu(x)}{\sigma(x)} \right| .$$

Then there exist positive constants C_1, C_2 and N_0 depending only on p such that for all $N > N_0$,

$$(1.2) \qquad C_1 N^{-1} \exp\{-\pi (\tfrac{2N}{p'})^{1/2}\} \le \delta_N \le C_2 N^{1/(2p')} \exp\{-\pi (\tfrac{N}{2p'})^{1/2}\}.$$

The rational functions of this paper are of the form μ/σ in (1.1) and they approximate f on $[-1,1]$ to within an error bounded by the right hand side of (1.2).

A typical approximation result of the present paper is the following:

THEOREM 1.2: *Let* f *and* g *satisfy the conditions of Thm.* 1.1, *and define* z_j *and* $B(z)$ *by*

$$(1.3) \qquad z_j = \frac{q^j-1}{q^j+1}, \quad B(z) = (1-z^2)\,\Pi_{j=-N}^{N} \frac{z-z_j}{1-z_j z}.$$

If q *is selected by the formula*

$$(1.4) \qquad q = \exp\{-\pi(\frac{p'}{2N})^{1/2}\}.$$

Then, for all integers $N > 0$,

$$(1.5) \qquad \max_{-1\le x\le 1} \left| f(x) - \Sigma_{j=-N}^{N} \frac{f(z_j)B(z)}{(z-z_j)B'(z_j)} \right| \le C_2 N^{1/(2p')} \exp\{-\pi(\frac{N}{2p'})^{1/2}\},$$

where C_2 *depends only on* p.

Due to their simplicity of construction and approximation properties, the rational function approximations of this paper play a similar role as the interpolation polynomials obtained by interpolation at the zeros of the Chebyshev polynomials play for polynomial approximation. In order to describe this role effectively, we return first to the case of Fourier series.

Let $R > 1$, and let A_R denote the annular region in the complex plane $\mathbb{C}$, $A_R = \{w \in \mathbb{C}: R^{-1} < |w| < R\}$, let F be analytic in A_R, and let c_j be determined from

$$(1.6) \qquad c_j = \frac{1}{2N+1} \Sigma_{k=0}^{2N} F(e^{i\theta_k})e^{-ij\theta_k}; \quad \theta_k = \frac{2k\pi}{2N+1}.$$

Then

$$(1.7) \qquad \max_{0 \leq \theta \leq 2\pi} |F(e^{i\theta}) - \Sigma_{j=-N}^{N} c_j e^{ij\theta}| = O(R^{-N}).$$

The bound on the right hand side is essentially best possible with regards to order, in that the number R cannot be replaced by a larger positive number regardless of how the c_j are chosen.

In (1.7) we now consider only those functions F for which $F(z) = F(1/z)$. Then we obtain a cosine polynomial approximation. The mapping

$$(1.8) \qquad z = \frac{1}{2}(w + \frac{1}{w})$$

transforms the annulus A_R onto the ellipse E_R with foci at $z = \pm 1$ and sum of semi-axes equal to R. Conversely, if $f(z)$ is analytic and uniformly bounded in E_R then we can use (1.8) to get a new function $F(w)$ analytic in A_R with Fourier series expansion $\Sigma_{k=-\infty}^{\infty} c_k e^{ik\theta}$, and where $c_{-k} = c_k$. If $T_N(x) = \cos N\theta$, where $x = \cos\theta$, and $x_k = \cos[(2k-1)\frac{\pi}{2N}]$, then

$$(1.9)\qquad \max_{-1\le x\le 1} \left| f(x) - \sum_{k=1}^{N} \frac{f(x_k)T_N(x)}{(x-x_k)T_N'(x_k)} \right| = O(R^{-N}),$$

where once again, the R in the $O(R^{-N})$ bound on the right hand side cannot be replaced by a larger number, regardless of how a polynomial of degree N-1 is chosen to approximate f on $[-1,1]$.

Hence, instead of finding the polynomial which best approximates f on $[-1,1]$, it is much easier to use the Chebyshev polynomial for which the interpolation points x_k are known explicitly to get an approximation which is nearly as good. The rational functions of this paper share this feature.

Notice that for the case of polynomial approximation above, we required a knowledge of a region of analyticity of f, a property which we can usually determine a priori in applications. Once we have identified such an ellipse E_R (resp. an annulus A_R) we can be certain that polynomial (rsp. Fourier polynomial) approximation will work very well on $[-1,1]$ (rsp. on $[0,2\pi]$). From the point of view of approximation in applications, we can thus identify functions analytic in E_R (rsp. A_R) with polynomials (rsp. Fourier polynomials) since they can be very accurately approximated with polynomials (rsp. Fourier polynomials) of low degree.

Unfortunately there is a drastic change in the rate of convergence of polynomial approximation in the case when the function to be approximated has a singularity on the interval of approximation, a situation often encountered in applications. For example, if $0 < \alpha < 1$, we have

$$(1.10) \qquad \max_{-1\le x\le 1} |(1-x^2)^\alpha - p_N(x)| \ge \frac{C}{N^\alpha}$$

where $p_N(x)$ is any polynomial of degree N in x and C is a constant independent of N. If $\alpha = 1/4$ we would have to take $N > 10^6$ to get 3 places of accuracy.

While for practical purposes functions with singularities on the interval of approximation cannot be identified with polynomials, there is, nevertheless, a class of functions with singularities on the interval of approximation which we describe in this paper, and which lends itself to accurate rational approximation. Such a class includes the functions which we can accurately approximate with polynomials and for practical purposes, we can identify this class with rational functions. For example, by Theorem 1.2 above, given an integer $N > 0$ there exists a rational function $p_{2N+2}(x)/q_{2N+1}(x)$ with p_{N+1} of degree $N+1$ in x and q_N of degree N in x, such that

$$(1.11) \qquad \max_{-1\le x\le 1} \left|(1-x^2)^\alpha - \frac{p_{2N+2}(x)}{q_{2N+1}(x)}\right| < C\, N^{\alpha/2}\exp(-\pi(\frac{\alpha N}{2})^{1/2}].$$

We remark that by identifying classes of functions which can be approximated accurately by rational functions, we are identifying classes of functions for which we can expect the Padé method, the Thiele algorithm or the Epsilon algorithm work well. We shall later in this paper illustrate this. For example, we would be able to tell a priori, that the Padé method used in [2] may be expected to be accurate.

Another practically important use of rational functions is in analytic continuation. For sake of illustration, let us momentarily return to the class of functions analytic and bounded in the ellipse E_R described above. Let us assume that f is known on $[-1,1]$, and that we want to evaluate f at the point $\frac{1}{2} + \frac{1}{4}(R + \frac{1}{R})$ in the ellipse. This can be done by means of the polynomial in (1.9), the rate of convergence of the error zero being $O(\rho^N)$, where $\rho = [(a+1)/2 + \sqrt{a+(a^2-3)/4}]/R$, $a = (R + R^{-1})/2$. On the other hand, if $p_N(x)$ is <u>any</u> polynomial approximation to $f(x) = c + (1-x^2)^{\alpha}$ on $[-\frac{1}{2},\frac{1}{2}]$ and we want to approximate $f(1) = c$ by evaluating $p_N(1)$, then we may expect $[f(1)-p_N(1)]$ to converge to zero very slowly, indeed, too slowly to be of any practical use. Since however, we may identify $f(x) = c + (1-x^2)^{\alpha}$ with a rational function for practical purposes, we can accurately evaluate $f(1)$ via a rational function, by using values of x on $[-\frac{1}{2},\frac{1}{2}]$ only.

As a more sophisticated example, let $u = u(x,y)$ be harmonic in the right-half plane, and assume that $u(0^+,y)$ is of class $\text{Lip}_\alpha (\alpha > 0)$ on a neighborhood of $y = 0$. It follows, then, that $u(x,0)$ is analytic and bounded on a sector with vertex at the origin and of class Lip_α on $[0,A]$ where $A > 1$ is arbitrary. That is, for practical purposes, we may identify $u(x,0)$ with a rational function, and we can accurately approximate $u(0,0)$ via a low degree rational function, by using values of $u(x,0)$ on (e.g.) the interval $[1,A]$.

In the cases when the condition of accurate approximation are satisfied, it is thus possible to do accurate analytic continuation all the way to the boundary of analyticity, via a relatively low degree rational function.

The Lip_α property of the function to be approximated is important from the point of view of applications; if f approaches zero too slowly in a neighborhood of a singularity, then it is necessary to choose the degree of the rational function to be very large, in order to achieve a desired accuracy. For example, for rational approximation on $[0,1]$, if $f(x) - f(1) \sim c/[\log(1-x)^{-1}]^\alpha$ as $x \to 1^-$ then it is just as difficult to approximate f on $[0,1]$ by a rational fun[illegible]n [illegible] it is to approximate $(1-x^2)^\alpha$ on $[-1,1]$ by a polynomial. We remark however, that this difficulty can often be remedied by means of a transformation. For example, if we set $x = 1-\exp(-z^{-1})$, we get $f(1-\exp(-z^{-1})) - f(1) \sim cz^{\alpha}$, $z \to 0$, and we can now approximate the new function of z defined as the interval $[0,\infty]$ by a rational function.

We mention that a rational function was previously constructed by the author [18] of the same degree as that in (1.5) for approximating f on $[-1,1]$, and moreover the error bound in [18] is the same as that on the right hand side of (1.5). However, whereas the interpolation points in [18] are the points

$$(1.12) \qquad \xi_j = k^{1/2}\, \mathrm{sn}[(2j-1)K/(2N);k]$$

the evaluation of the ξ_j is more difficult than the evaluation of the z_j in (1.3).

The same points z_j defined in (1.3) were also used by Peaceman and Rachford [13] to approximate the points ξ_j in (1.12) in their alternating direction iterative method for obtaining approximate solutions to elliptic partial differential equations.

For many problems of rational approximation one does not have analyticity in the unit disc U, but rather in a smaller region $\mathcal{D}_d^1$ (see Figure 2.1 in Sec. 2) and we have therefore also considered this case. Although our error bounds in this case are not as small as the sinc bounds, we suspect that the rationals of this paper are as accurate as the sinc approximations for the same problem, and that it may be possible to improve the bounds of this paper in the case when $0 < d < \pi/2$.

Notice that if N is replaced by 4N in the rational function in (1.5) then every second interpolation point in the "4N" rational case is the same as the interpolation point in the "N" rational case, and the "4N" rational approximation has roughly twice as many correct significant figures as the "N" rational approximation. This result is of practical value, particularly when a user is unable to determine a region $\mathfrak{D}_d^1$ (Figure 2.1).

Let us now briefly describe the layout of the paper.

In Sec. 2 we give precise statements and proofs of the results (a), (b), (c) and (d) stated at the beginning of this section. These proofs would ordinarily be lengthy, and for this reason some of the details are carried out in appendix A and B.

In Sec. 3 we illustrate connections of the results with the well-known approximation algorithms, the Thiele, or ρ-algorithm, the Epsilon algorithm (ϵ-algorithm) and the Padé method. In view of the results of Sec. 2, we are able to determine a priori, when we can expect these algorithms to work.

In Sec. 4, we prove Theorem 1.1 above. While the exact optimal rate of convergence of rational approximation is not known, we conjecture that, in the notation of (1.1),

$$(1.13) \qquad \inf_{\mu\in\mathbf{P}_N,\ \sigma\in\mathbf{P}_N}\ \sup_{g\in H_p(U)\,.\ \|g\|_p=1}\ \sup_{-1<x<1}\left|f(x)-\frac{\mu(x)}{\sigma(x)}\right|^{(p'/N)^{\frac{1}{2}}} \to e^{-\pi}$$

as $N \to \infty$.

In Appendix A we study bounds on rational functions related to (1.5). The Jacobi theta functions turn out to be most convenient for this purpose, since. while it is possible to obtain similar results via the approximate integration of the function $F(z;t) = t^{-1}\log|(z+t)/(z-t)|$ over $0 \leq t \leq \infty$ via the trapezoidal (resp. midordinate) rule evaluated at the points q^j (rsp. $q^{j-1/2}$), $j = 0,\pm 1,\pm 2,\ldots$, and using the concavity of this function for (fixed $z\in(0,\infty)$) as a function of t, it is possible to achieve more accurate error bounds via the theta functions since it is possible to get exact bounds via known properties of the theta functions. However, while we use elliptic functions to obtain our results, the final results are independent of elliptic functions.

In Appendix B we obtain accurate bounds on contour integrals encountered in the proofs in Sec. 2.

We close this section with a few historical remarks.

Stietjes [20] seems to be the first to identify classes of function which may be approximated via truncated forms of continued fractions. These functions are expressible in the form

$$(1.14) \qquad F(z) = \int_a^b \frac{d\mu(t)}{t-z}$$

and the continued fraction expression obtainable via this representation converges uniformly in any closed region of the complex plane which does not contain the interval [a,b] (see [7]). Unfortunately given a function F it is not possible to easily check in applications whether or not F has a representation of the form (1.14).

In [6] Gautschi gives an excellent summary of the use of rational functions in numerical analysis. It has long been suspected and verified in ad hoc cases that a rational function can do a better job of approximation in a neighborhood of a singularity than a polynomial. The first quantitative result as to exactly how much better a rational function can be than a polynomial was obtained by Newman [10,11]. Renewed interest has developed in rational functions since Newman's result. The error of the rational functions of this paper have the $O(e^{-cn^{1/2}})$ rate of convergence which was obtained first by Newman. Saff and Varga (see [14,22]) have obtained many beautiful results demonstrating the superior power of rational functions. Also of interest is the idea of Ganelius [5], for using the Greens function of a region of analyticity to obtain rational approximations; indeed the rational functions of this paper have this property. For the case of rational approximation on a finite or semi-infinite interval, the poles of the rational functions of this paper lie on the real line outside of the interval as is the case for best approximation of Stietjes transforms--see Borwein [3].

2. RATIONAL APPROXIMATION WITH ERROR BOUNDS

This section contains the main approximation theorem of the paper. While the proofs are complete, we use results derived in the appendix in order to shorten these proofs.

At the outset we cover in detail, the case of rational approximation on the interval [-1,1]. These results are then extended to the case of rational approximation with a rational function of a variable φ over a contour P, where φ is a one-one transformation of P onto the interval [-1,1]. We then use this generalized result to obtain two types of rational approximations over the interval $[0,\infty]$ and one over $[-\infty,\infty]$.

Rational approximation on the interval [-1,1] is effective when applied to a certain class of functions that is analytic in the region

$$\mathfrak{D}_d^1 = \{\zeta \in \mathbb{C}: |\arg\frac{1+\zeta}{1-\zeta}| < d\}, \quad 0 < d \le \pi, \tag{2.1}$$

where $\mathbb{C}$ denotes the complex plane. In the case when $d = \pi/2$, $\mathfrak{D}_d^1$ is the unit disc. When $0 < d < \pi/2$, $\mathfrak{D}_d^1$ is the intersection of the two discs

$$U_{1,2} = \{\zeta \in \mathbb{C}: |\zeta \pm i \cot d| \le |\csc d|\} \tag{2.2}$$

while if $\pi/2 \le d < \pi$, $\mathfrak{D}_d^1$ is the union of the two

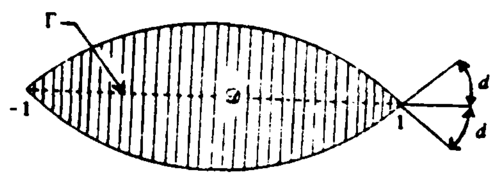

Fig. 2.1. The Region $\mathfrak{D}_d^1$ of Eq. (2.1).

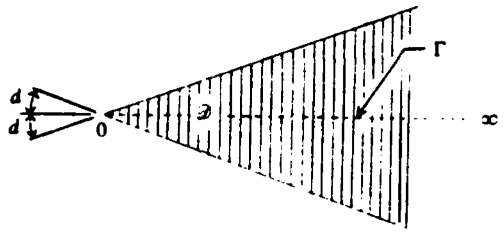

Fig. 2.2. The Region $\mathfrak{D}_d^2$ of Eq. (2.3).

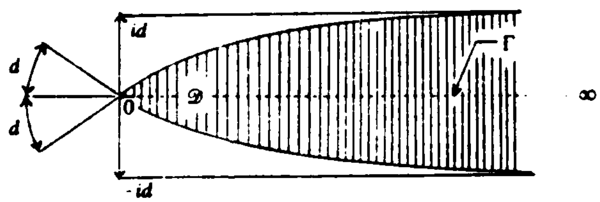

Fig 2.3. The Region $\mathfrak{D}_d^3$ of Eq. (2.4).

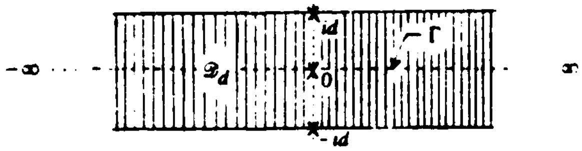

Fig. 2.4. The Region $\mathfrak{D}_d^4$ of Eq. (2.5).

discs (2.2). We shall obtain rational approximations for functions analytic in $\mathfrak{D}_d^1$, and also for functions analytic on the regions.

$$(2.3) \qquad \mathfrak{D}_d^2 = \{z \in \mathbb{C}: |\arg z| < d\},$$

$$(2.4) \qquad \mathfrak{D}_d^3 = \{z \in \mathbb{C}: |\arg \sinh z| < d\},$$

$$(2.5) \qquad \mathfrak{D}_d^4 = \{z \in \mathbb{C}: |\mathrm{Im}\, z| < d\}.$$

These regions are illustrated in Figs. 2.1-2.4.

2.1 Rational Approximation on [-1,1].

We describe two typical situations of rational approximation on the interval [-1,1]. The conditions in the first case are of theoretical interest, particularly when $\mathfrak{D}_d$ is the unit disc, while the conditions in the second case can be readily tested in applications.

Given f analytic in $\mathfrak{D}_d^1$, we define F by

$$(2.6) \qquad F(\zeta) = \frac{f(\zeta)}{1-\zeta^2}.$$

Assumption 2.1a: Let f be analytic in $\mathfrak{D}_d^1$, let F be defined by (2.6), and for some p in the range $1 < p < \infty$, let

$$(2.7) \qquad \|F\|_p \equiv \lim_{\delta \to d} \left(\frac{1}{2\pi} \int_{\partial \mathfrak{D}_\delta^1} |F(\zeta)|^p |d\zeta|\right)^{1/p} < \infty.$$

Let σ be either 0 or $\frac{1}{2}$, let $0 < q < 1$, let ζ_j be defined by

$$(2.8) \qquad \zeta_j = \frac{q^{j-\sigma}-1}{q^{j-\sigma}+1}$$

and corresponding to a positive integer N, let $B(\zeta)$ be defined by

$$(2.9) \qquad B(\zeta) = \Pi_{j=-N+2\sigma}^{N} \frac{\zeta-\zeta_j}{1-\zeta_j\zeta} .$$

Theorem 2.1a: Let Assumption 2.1a be satisfied, let $0 < d \leq \pi/2$, and define η by

$$(2.10) \qquad \eta(\xi) = f(\xi) - \Sigma_{j=-N+2\sigma}^{N} \frac{(1-\xi^2)B(\xi)f(\zeta_j)}{(\xi-\zeta_j)(1-\zeta_j^2)B'(\zeta_j)}$$

where $B(\zeta)$ is defined in (2.9). If q is selected by the formula

$$(2.11) \qquad q = \exp\{-\pi(\frac{p'}{2N})^{1/2}\},$$

where $p' = p/(p-1)$, then there exists a constant C depending only on p, such that for $-1 \leq \xi \leq 1$,

$$(2.12) \qquad |\eta(\xi)| \leq CN^{1/(2p')}\exp\{-d(\frac{2N}{p'})^{1/2}\}\|F\|_p.$$

Proof: It is readily seen that $\eta(\xi)$ in (2.10) also has the representation

$$(2.13) \qquad \eta(\xi) = \frac{(1-\xi^2)B(\xi)}{2\pi i} \int_{\partial\mathfrak{D}_d^1} \frac{f(\zeta)d\zeta}{(\zeta-\xi)(1-\zeta^2)B(\zeta)}.$$

Now, by Thm. A.10 (i), the function $|B(\zeta)|^{-1}$ satisfies the inequality

$$(2.14) \qquad \max_{\zeta \in \partial\mathfrak{D}_d} |B(\zeta)|^{-1} \leq \exp\{\frac{\pi(\frac{\pi}{2}-d) + \epsilon(d,q)}{\log \frac{1}{q}}\}$$

where, by (A.36), we have, for $0 < d \leq \pi/2$,

$$(2.15) \qquad \frac{\epsilon(d,q)}{\log \frac{1}{q}} \leq \frac{\pi^2}{\log \frac{1}{q}} \exp(-\frac{2\pi d}{\log \frac{1}{q}}) \leq \frac{\pi}{2ed}.$$

Hence, setting

$$(2.16) \qquad A = \exp(\frac{\pi}{2d} e^{-1}),$$

taking absolute values of each term in (2.13) and using (2.6), we get

$$(2.17) \qquad |\eta(\xi)| \leq \frac{(1-\xi^2)|B(\xi)|}{2\pi} A \exp\{\frac{\pi(\frac{\pi}{2}-d)}{\log \frac{1}{q}}\} \int_{\partial\mathfrak{D}_d} \frac{|F(\zeta)|}{|\zeta-\xi|} |d\zeta|$$

$$(2.18) \qquad \leq A(1-\xi^2)|B(\xi)| \exp\{\frac{\pi(\frac{\pi}{2}-d)}{\log \frac{1}{q}}\} [G(p',d,\xi)]^{\frac{1}{p'}} \|F\|_p$$

where (2.18) is obtained after applying Holder's inequality to the integral on the right hand side of (2.17) and then using the notation of Eq. (B.18) of Appendix B. By applying Thm. B.1 to bound $G(p',d,\xi)$, we therefore get

$$(2.19) \qquad |\eta(\xi)| \leq A\, 2^{1/p'} (\sin d)^{(p'-1)/(2p')} [\frac{\Gamma(p'-1)}{\Gamma(\frac{p'}{2})\Gamma(\frac{p'}{2})}]^{1/p'} \|F\|_p.$$

(2.19 cont.)
$$\cdot (1-\xi^2)^{1/p'} |B(\xi)| \exp\{\frac{\pi(\frac{\pi}{2}-d)}{\log \frac{1}{q}}\}.$$

We shall now attempt to obtain a uniform bound on $(1-\xi^2)^{1/p'} |B(\xi)|$. To this end, let us select a number ξ_1 such that

$$(2.20) \qquad 0 < q^{N+1/2-\sigma} \leq \xi_1 < 1$$

and let us define $I(\xi_1)$ by

$$(2.21) \qquad I(\xi_1) = \{x: \frac{\xi_1 - 1}{\xi_1 + 1} \leq x \leq \frac{1-\xi_1}{1+\xi_1}\}.$$

Then, by Thm. (A.10)

$$(2.22) \qquad \max_{\xi \in I(\xi_1)} |B(\xi)| \leq b(N,q,\xi_1)$$

where

$$(2.23) \qquad b(N,q,\xi_1) = 2 \exp\{-\frac{\pi^2}{2 \log \frac{1}{q}}(1-q^N/\xi_1)\}.$$

Since $|B(\xi)| \leq 1$ on $[-1,1]$, we have

$$(2.24) \qquad (1-\xi^2)^{1/p'} |B(\xi)| \leq \begin{cases} b(N,q,\xi_1) & \text{if } \xi \in I(\xi_1) \\ \left(\frac{4\xi_1}{(1+\xi_1)^2}\right)^{1/p'} & \text{if } \xi \in [-1,1]-I(\xi_1). \end{cases}$$

We now define q by (2.11), ξ_1 by

$$(2.25) \qquad \xi_1 = N^{1/2} q^N$$

to get (2.12), in which we may take C to be

$$(2.26)\qquad C = 2^{3/p'}(\sin d)^{\frac{p'-1}{2p'}}\left\{\frac{\Gamma(p'-1)}{\Gamma(p'/2)\Gamma(p'/2)}\right\}^{\frac{1}{p'}}\exp\{\pi[(2ed)^{-1}+(2p')^{-\frac{1}{2}}]\}.$$

Remark (i): If $d = \pi/2$, the result (2.12) remains the same, except that the constant C may be taken to be

$$(2.27)\qquad C = 2^{1+1/p'}\exp\{\pi(2p')^{-1/2}\}\left\{\frac{\Gamma(p'-1)}{\Gamma(p'/2)\Gamma(p'/2)}\right\}^{1/p'}.$$

Remark (ii): From (2.12), the dominant rate of convergence of the error to zero as $N \to \infty$ is

$$(2.28)\qquad r_1 = \exp\{-d(\tfrac{2N}{p'})^{1/2}\}.$$

The corresponding rate for sinc approximation is

$$(2.29)\qquad r_2 = \exp\{-(\tfrac{\pi d}{p'}N)^{1/2}\}.$$

This rate r_2 converges to zero more rapidly than r_1 if $0 < d < \pi/2$, and $r_2 = r_1$ if $d = \pi/2$. The author originally expected these rates to be the same for all d in the range $0 < d \leq \pi/2$, but we have so far been unable to obtain the sharper bound of (2.29).

Remark (iii): It would be interesting to extend Thm. 2.1a to the case $\pi/2 < d \leq \pi$. Thm. A.10 applies to this case; it shows moreover that we must more carefully bound the contour integral (2.13), taking advantage of the fact that whereas $|B(\zeta)|^{-1}$ is close to 1 in neighborhoods of $\zeta = \pm 1$, it is close to $2 \exp\{\frac{\pi(\pi/2-d)}{\log 1/q}\}$ on the part of $\partial\mathfrak{D}_d^1$ outside of the neighborhoods of ± 1. We have not done this, due to the added complexity of the proofs. However, such an approach may also lead to an improvement of the bound (2.12) for the case when $0 < d < \pi/2$.

Remark (iv): The Assumption 2.1a is not satisfied if $f(\xi)$ does not approach zero as $\xi \to \pm 1$. However, if g is analytic and bounded in $\mathfrak{D}_d$ and is of class $\mathrm{Lip}_\alpha(\bar{\mathfrak{D}}_d)$, where $\bar{\mathfrak{D}}_d$ denotes the closure of $\mathfrak{D}_d$, then f defined by

$$(2.30) \qquad f(\xi) = g(\xi) - \frac{1-\xi}{2} g(-1) - \frac{1+\xi}{2} g(1)$$

satisfies Assumption 2.1a, for $p < 1/(1-\alpha)$. After obtaining a rational approximation for f we add the linear term $\frac{1}{2}(1-\xi)g(-1) + \frac{1}{2}(1+\xi)g(1)$ to get a rational approximation for g.

Rather than testing whether or not Assumption 2.1a is satisfied, it is often simpler in practice to check whether or not the following assumption is satisfied. All of the above remarks, albeit after some obvious modifications, apply also to this case.

Assumption 2.1b: *Let f be analytic in $\mathfrak{D}_d$ and let*

$$(2.31)\qquad |f(\zeta)| \le c|(1+\zeta)^{\alpha}(1-\zeta)^{\beta}|$$

for all ζ in $\mathfrak{D}_d^1$, where c, α and β are positive numbers, and $0 < \alpha < 1$, $0 < \beta < 1$.

Theorem 2.1b: Let Assumption 2.1b be satisfied, and let $0 < d \le \pi/2$. Let $\gamma = \min(\alpha,\beta)$, $\delta = \max(\alpha,\beta)$, and corresponding to a positive integer n, let q be defined by

$$(2.32)\qquad q = \exp\{-\pi(2\gamma n)^{-1/2}\}.$$

If $\gamma = \alpha$, let M and N be defined by

$$(2.33)\qquad \begin{aligned} M &= n \\ N &= [\tfrac{\alpha}{\beta}n] \end{aligned}$$

while if $\gamma = \beta$, let M and N be defined by

$$(2.34)\qquad \begin{aligned} M &= [\tfrac{\beta}{\alpha}n] \\ N &= n. \end{aligned}$$

Let ζ_j be defined by (2.8), and let $B(\zeta)$ and $\eta(\xi)$ be defined by

$$(2.35)\qquad B(\zeta) = \Pi_{j=-M+2\sigma}^{N} \frac{\zeta-\zeta_j}{1-\zeta_j\zeta}$$

and

$$(2.36)\qquad \eta(\xi) = f(\xi) - \Sigma_{j=-M+2\sigma}^{N} \frac{(1-x^2)B(\xi)f(\zeta_j)}{(\xi-\zeta_j)(1-\zeta_j^2)\ B'(\zeta_j)}.$$

<u>Then there exists a constant</u> C <u>depending only on</u> c, α, β <u>and</u> d, <u>such that for all</u> ξ <u>in</u> $[-1,1]$,

$$(2.37)\qquad |\eta(\xi)| \le C\ n^{\delta/2} \exp\{-d(2\gamma n)^{1/2}\}.$$

<u>Proof:</u> The identity (2.13) now also defines the $\eta(\xi)$ of (2.36), provided that $B(\zeta)$ is defined by (2.35). Taking absolute values, replacing $|f(\zeta)|$ by the right hand side of (2.31), and using (2.14)-(2.16), we get

$$(2.38)\qquad |\eta(\xi)| \le Ac(1-\xi^2)|B(\xi)|H(\alpha,\beta,d,\xi)\exp\{\frac{\pi(\frac{\pi}{2}-d)}{\log\frac{1}{q}}\},$$

where $H(\alpha,\beta,d,\xi)$ is defined by Eq. (B.19) of Appendix B. Using the result of Thm. B.2, we now get

$$(2.39)\qquad |\eta(\xi)| \le A\ c\ C_1(1+\xi)^{\alpha}(1-\xi)^{\beta}|B(\xi)|\exp\{\frac{\pi(\frac{\pi}{2}-d)}{\log\frac{1}{q}}\},$$

where the constant C_1 depends only on α, β and d.

We now proceed to obtain a uniform bound on $(1+\xi)^{\alpha}(1-\xi)^{\beta}|B(\xi)|$. To this end let ξ_1 and ξ_2 be selected such that

$$(2.40)\qquad \begin{aligned} 0 &< q^{M+1/2-\sigma} \le \xi_1 < 1 \\ 0 &< q^{N+1/2-\sigma} \le \xi_2 < 1 \end{aligned}$$

and define $I(\xi_1, \xi_2)$ by

$$(2.41) \qquad I(\xi_1,\xi_2) = \{\xi : \frac{\xi_1 - 1}{\xi_1 + 1} \le \xi \le \frac{1-\xi_2}{1+\xi_2}\}.$$

Then, by Thm. A.10,

$$(2.42) \qquad \max_{\xi \in I(\xi_1,\xi_2)} |B(\xi)| \le b(M,N,\xi_1,\xi_2)$$

where

$$(2.43) \qquad b(M,N,\xi_1,\xi_2) \le 2\exp\{- \frac{\pi^2}{2 \log \frac{1}{q}}[1 - \frac{q^M}{2\xi_1} - \frac{q^N}{2\xi_2}]\}.$$

We therefore have, since $|B(x)| \le 1$ if $-1 \le x \le 1$,

$$(2.44) \qquad (1+\xi)^{\alpha}(1-\xi)^{\beta}|B(\xi)| \le \begin{cases} 2^{2\alpha+\beta}\xi_1^{\alpha} & \text{if } -1 \le \xi \le \frac{\xi_1 - 1}{\xi_1 + 1} \\ b(M,N,\xi_1,\xi_2) & \text{if } \xi \in I(\xi_1,\xi_2) \\ 2^{\alpha+2\beta}\xi_2^{\beta} & \text{if } \frac{1-\xi_2}{1+\xi_2} \le \xi \le 1. \end{cases}$$

Now q is given by (2.32) and we consider first the case when $\gamma = \alpha = \min(\alpha,\beta)$. In this case $M = n$, $N = [\alpha n/\beta]$. If we furthermore select ξ_1 and ξ_2 by the formulas $\xi_1 = q^M M^{1/2}$, $\xi_2 = q^N N^{1/2}$, then we get the uniform bound (2.37).

The argument in the case when $\gamma = \beta$ is similar, and we omit it.

2.2 Rational Approximation on a Contour.

Let $\mathfrak{D}$ be a simply connected domain with boundary $\partial\mathfrak{D}$, let a and b $(b \neq a)$ belong to $\partial\mathfrak{D}$, and let φ be a conformal map of $\mathfrak{D}$ onto the region $\mathfrak{D}_d^1$ (Eq. (2.1)) such that $\varphi(a) = -1$, $\varphi(b) = 1$. Let $\psi = \varphi^{-1}$ denote the inverse map, and define Γ by

$$(2.45) \qquad \Gamma = \{\psi(\zeta) : -1 \leq \zeta \leq 1\}.$$

Let z_j denote the point

$$(2.46) \qquad z_j = \psi(\zeta_j),$$

where the ζ_j are defined as in (2.8). Let f be analytic in $\mathfrak{D}$ and define $\mathfrak{F}$ by

$$(2.47) \qquad \mathfrak{F}(z) = \frac{f(z)}{1-\varphi(z)^2}\,.$$

Let $\mathfrak{B}(z)$ be defined by

$$(2.48) \qquad \mathfrak{B}(z) = \Pi_{j=-M+2\sigma}^{N} \frac{\varphi(z)-\zeta_j}{1-\zeta_j\varphi(z)}$$

where σ is defined as in (2.8).

Let one of the following two assumptions be satisfied:

<u>Assumption 2.2a</u>: <u>Let</u> $\mathfrak{F}$ <u>be defined by</u> (2.47), <u>and for some</u> $p > 1$, <u>let</u>

$$(2.49) \qquad \|\mathfrak{F}\| \equiv \liminf_{C\subseteq\mathfrak{D},\, C\to\partial\mathfrak{D}} \left(\frac{1}{2\pi}\int_C |\mathfrak{F}(z)|^p |\varphi'(z)||dz|\right)^{1/p} < \infty.$$

Assumption 2.2b: Let f be analytic in $\mathfrak{D}$, and for all $z \in \mathfrak{D}$, let

$$(2.50) \qquad |f(z)| \le A|1-\varphi(z)|^\alpha |1+\varphi(z)|^\beta$$

where A, α and β are numbers such that $A > 0$, $0 < \alpha < 1$, $0 < \beta < 1$.

Proceeding as in the previous section, we set

$$(2.51) \qquad \eta(x) = \frac{[1-\varphi(x)^2]\mathfrak{B}(x)}{2\pi i}\int_{\partial\mathfrak{D}} \frac{f(z)\varphi'(z)dz}{[\varphi(z)-\varphi(z)][1-\varphi(z)^2]\mathfrak{B}(z)}, x \in \Gamma,$$

to get

$$(2.52) \qquad \eta(x) = f(x) - \Sigma_{j=-M+2\sigma}^{N} \frac{[1-\varphi(x)^2]\mathfrak{B}(x)\varphi'(z_j)}{[\varphi(x)-\zeta_j][1-\zeta_j^2]\mathfrak{B}'(z_j)} .$$

The sum on the right hand side is a rational function in the variable $\varphi(x)$.

Theorem 2.2a: Let Assumption 2.2a be satisfied, let $0 < d \le \pi/2$, let $M = N$ in (2.48) and (2.52), and let Γ be defined by (2.47). If q is defined by

$$(2.53) \qquad q = \exp\{-\pi(\frac{p'}{2N})^{1/2}\}$$

where $p' = p/(p-1)$, then there exists a constant C

depending only on p such that for all x on Γ

$$(2.54)\qquad |\eta(x)| \le CN^{1/2p'}\exp\{-d(\tfrac{2N}{p'})^{1/2}\}\|\mathfrak{F}\|_p.$$

Proof: If we set $\varphi(z) = \zeta$ in (2.51), we get (2.7) with $\mathfrak{F}(\psi(z)) = F(z)$; if we set $\varphi(\xi) = x$ and $\varphi(z) = \zeta$ in (2.51) we get (2.13). Hence the proof is identical to the proof of Thm. 2.1a.

The proof of the following theorem is also similar to the proof of Thm. 2.1b, in view of the above remarks.

Theorem 2.2b: Let Assumption 2.2b be satisfied, and let $0 < d \le \pi/2$. Let $\gamma = \min(\alpha,\beta)$, $\delta = \max(\alpha,\beta)$, and corresponding to a positive integer n, let q be defined by

$$(2.55)\qquad q = \exp\{-\pi(2\gamma n)^{1/2}\}.$$

If $\gamma = \alpha$, let M and N be defined by

$$(2.56)\qquad \begin{aligned} M &= n \\ N &= [\tfrac{\alpha}{\beta}n] \end{aligned}$$

while if $\gamma = \beta$, let M and N be defined by

$$(2.57)\qquad \begin{aligned} M &= [\tfrac{\beta}{\alpha}n] \\ N &= n. \end{aligned}$$

Let ζ_j be defined by (2.8), z_j by (2.46), Γ by (2.45) and $\mathfrak{B}(z)$ and $\eta(x)$ by

$$(2.58) \qquad \mathfrak{B}(z) = \Pi_{j=-M+2\sigma}^{N} \frac{\varphi(z)-\zeta_j}{1-\zeta_j\varphi(z)}$$

and

$$(2.59) \qquad \eta(x) = f(x) - \Sigma_{j=-M+2\sigma}^{N} \frac{[1-\varphi(x)^2]\mathfrak{B}(x)f(z_j)\varphi'(z_j)}{[\varphi(x)-\zeta_j](1-\zeta_j^2)\mathfrak{B}'(z_j)} .$$

Then there exists a constant C depending only on α and β such that for all $x \in \Gamma$,

$$(2.60) \qquad |\eta(x)| \le C\, n^{\delta/2} \exp\{-d(2\gamma n)^{1/2}\}.$$

2.3 Rational Approximation on $[0,\infty]$; the Non-oscillatory Case.

In this case, the region $\mathfrak{D}$ of analyticity is the sector $\mathfrak{D}_d^2$ of Eq. (2.3). The function φ and the inverse function ψ of the previous section are

$$(2.61) \qquad z = \psi(\zeta) = \frac{1+\zeta}{1-\zeta} \Leftrightarrow \zeta = \varphi(z) = \frac{z-1}{z+1}.$$

Hence corresponding to the points ζ_j in (2.8), the points z_j are

$$(2.62) \qquad z_j = q^{j-\sigma} \qquad (\sigma = 0 \text{ or } 1/2).$$

Thus the product (2.48) becomes

$$(2.63) \qquad \mathfrak{B}(z) = \Pi_{j=-M+2\sigma}^{N} \frac{z-q^{j-\sigma}}{z+q^{j-\sigma}} .$$

Corresponding to f analytic in $\mathfrak{D}_d^2$ the function $\mathfrak{F}$ in (2.47) is now defined by

$$(2.64) \qquad \mathfrak{F}(z) = \frac{(1+z)^2}{4z} f(z);$$

the condition (2.49) thus becomes

$$(2.65) \qquad \lim_{\tau \to 0^+} \left(\frac{1}{2\pi} \int_{\partial \mathfrak{D}_d^2(\tau)} |\mathfrak{F}(z)|^p \frac{|dz|}{|1+z|^2}\right)^{1/p} < \infty, \; 1 < p < \infty,$$

where $\mathfrak{D}_d^2(\tau) = \{z \in \mathbb{C}: |\arg(z-\tau)| < d\}$, while the condition (2.50) becomes

$$(2.66) \qquad |f(z)| \leq A|1+z|^{-\alpha-\beta}|z|^{\beta}.$$

The error η, i.e., the difference between f and the rational function thus becomes

$$(2.67) \qquad \eta(x) = f(x) - \frac{x\mathfrak{B}(x)}{1+x} \Sigma_{j=-M+2\sigma}^{N} \frac{(1+q^{j-\sigma}) f(z_j)}{q^{j-\sigma}(x-z_j)\mathfrak{B}'(z_j)}.$$

It thus follows that if f is analytic in $\mathfrak{D}_d^2$, and (2.65) is satisfied we take $M = N$ in (2.63) and (2.67), and choose q as in (2.53) to bound η on $[0,\infty]$ according to (2.54). If f is analytic in $\mathfrak{D}_d^2$ and (2.66) is satisfied, then by choosing q as in (2.55), we bound η on $[0,\infty]$ according to (2.60).

2.4 Rational Approximation on $[0,\infty]$; the Oscillatory Case.

In this case the region $\mathfrak{D}$ of analyticity is the region $\mathfrak{D}_d^3$ of Eq. (2.4). The mapping functions φ and ψ are defined by

$$\varphi(z) = \frac{\sinh z - 1}{\sinh z + 1}, \quad \psi(\zeta) = \frac{1 + \sinh^{-1}(\frac{1+\zeta}{1-\zeta})}{1 - \sinh^{-1}(\frac{1+\zeta}{1-\zeta})}. \tag{2.68}$$

We see therefore from (2.63) that $\mathfrak{B}(z)$ takes the form

$$\mathfrak{B}(z) = \Pi_{j=-M+2\sigma}^{N} \frac{\sinh z - q^{j-\sigma}}{\sinh z + q^{j-\sigma}}, \tag{2.69}$$

i.e., the points z_j are given by

$$z_j = \sinh^{-1}(q^{j-\sigma}) = \log\{q^{j-\sigma} + \sqrt{1+q^{2j-\sigma}}\}. \tag{2.70}$$

If f is analytic in $\mathfrak{D}_d^3$, the corresponding function $\mathfrak{F}$ is defined by

$$\mathfrak{F}(z) = \frac{(1 + \sinh z)^2}{4 \sinh z} f(z). \tag{2.71}$$

The condition (2.49) becomes

$$\|\mathfrak{F}\|_p = \lim_{\tau\to 0^+}\left(\frac{1}{2\pi} \int_{\partial\mathfrak{D}_d^3(\tau)} |\mathfrak{F}(z)|^p \frac{|2 \cosh z|}{|1+\sinh z|^2} |dz|\right)^{\frac{1}{p}} < \infty, \quad 1 < p < \infty, \tag{2.72}$$

where

$$\mathfrak{D}_d^3(\tau) = \{z \in \mathbb{C}: |\arg \sinh(z-\tau)| < d\},$$

and the following condition is equivalent to (2.50):

$$(2.73) \qquad |f(z)| \le A|z|^{\alpha}|e^{-\beta z}|, \qquad z \in \mathfrak{D}_d^3.$$

The difference, η, between f and the rational function becomes

$$(2.74) \qquad \eta(x) = f(x) - \frac{\sinh x}{1+\sinh x}\mathfrak{B}(x)\, \Sigma_{j=-M+2\sigma}^{N} \frac{1+q^{j-\sigma}}{q^{j-\sigma}} \frac{f(z_j)\sqrt{1+q^{2(j-\sigma)}}}{(\sinh x - z_j)\mathfrak{B}'(z_j)}$$

Either Thm. 2.2a or Thm 2.2b may now be stated to bound $\eta(x)$ on $[0,\infty]$.

2.5 Rational Approximation on $[-\infty,\infty]$ Via a Rational Function of e^x.

The region $\mathfrak{D}$ of analyticity is the strip $\mathfrak{D}_d^4$ defined in Eq. (2.5). The function φ and ψ become

$$(2.75) \qquad \varphi(z) = \frac{e^z-1}{e^z+1}, \qquad \psi(\zeta) = \log\frac{1+\zeta}{1-\zeta}.$$

The product $\mathfrak{B}(z)$ now takes the form

$$(2.76) \qquad \mathfrak{B}(z) = \Pi_{j=-M+2\sigma}^{N} \frac{e^z-q^{j-\sigma}}{e^z+q^{j-\sigma}},$$

i.e. the points z_j are given by

$$(2.77) \qquad z_j = (j-\sigma)\log q.$$

Let f be analytic in $\mathfrak{D}_d^4$. Then the corresponding function $\mathfrak{F}$ of (2.47) is defined by

$$(2.78) \qquad \mathfrak{F}(z) = \frac{(1+e^z)^2}{4e^z} f(z).$$

The condition (2.49) now becomes

$$(2.79) \qquad \|\mathfrak{F}\|_p = \lim_{\tau \to d^-} \left(\int_{\partial \mathfrak{D}(\tau)} |\mathfrak{F}(z)|^p \left| \frac{2e^z}{(1+e^z)^2} \right| |dz| \right)^{1/p} \leq \infty, \quad p > 1.$$

The following condition is equivalent to (2.50):

$$(2.80) \qquad |f(z)| \leq \begin{cases} A \exp(\alpha \operatorname{Re} z), & z \in \mathfrak{D}_d^4, \quad \operatorname{Re} z \leq 0 \\ A \exp(-\beta \operatorname{Re} z), & z \in \mathfrak{D}_d^4, \quad \operatorname{Re} z \geq 0. \end{cases}$$

The difference between f and the rational function now becomes

$$(2.81) \qquad \eta(x) = f(x) - \frac{e^x}{1+e^x} \mathfrak{B}(x) \sum_{j=-M+2\sigma}^{N} \frac{(1+q^{j-\sigma}) f(z_j)}{(e^x - q^{j-\sigma}) \mathfrak{B}'(z_j)}.$$

We may now bound $\eta(z)$ on $(-\infty, \infty)$ via either Thm. 2.2a or Thm. 2.2b.

3. IMPLICATIONS AND APPLICATIONS.

In this section we study the connection of the results of the previous section with the Thiele algorithm, the epsilon algorithm, and the Padé method.

3.1 The Thiele Algorithm.

The Thiele, or ρ algorithm for interpolating f at $m+1$ distinct points $x_0, x_1, \ldots, x_m$ is described as follows.

Define ρ_i^j by

$$
(3.1)\qquad
\begin{cases}
\rho_0^j = f(x_j) & j = 0,1,\ldots,m \\[2ex]
\rho_1^j = \dfrac{x_{j+1} - x_j}{\rho_0^{j+1} - \rho_0^j} & j = 0,1,\ldots,m-1 \\[2ex]
\rho_i^j = \dfrac{x_{i+j} - x_i}{\rho_{i-1}^{j+1} - \rho_{i-1}^j} + \rho_{i-2}^{j+1} & \begin{array}{l} j = 0,1,\ldots,m-i \\ i = 2,3,\ldots,m. \end{array}
\end{cases}
$$

Then the rational function $r(x)$ which interpolates the data $\{x_j, f(x_j)\}_{j=0}^m$ is given by the continued fraction representation

$$
(3.2)\qquad r(x) = \rho_0^0 + \frac{x - x_0 |}{| \rho_1^0} + \frac{x - x_1 |}{| \rho_2^0 - \rho_0^0} + \cdots + \frac{x - x_m |}{| \rho_m^0 - \rho_{m-2}^0} .
$$

The function $r(x)$ has the form

$$
(3.3)\qquad r(x) = \frac{p_n(x)}{q_n(x)}
$$

if $m = 2n$, where p_n and q_n are polynomials of degree n in x, and it has the form

$$(3.4) \qquad r(x) = \frac{p_{n+1}(x)}{q_n^*(x)}$$

if $m = 2n + 1$. Furthermore, if $m = 2n$, then

$$(3.5) \qquad r(x) = \frac{\rho_{2n}^{0} x^n + c_1 x^{n-1} + \ldots + c_n}{x^n + d_1 x^{n-1} + \ldots + d_n} .$$

That is

$$(3.6) \qquad \rho_{2n}^{0} = \lim_{x \to \pm\infty} r(x),$$

so that the Thiele algorithm provides an excellent method of carrying out analytic continuation.

For example, if f is analytic and bounded in the region $\mathfrak{D}_d^2$ of Eq. (2.3) and if f is of class Lip_α $(\alpha > 0)$ $[x_0, \infty]$, where $x_0 \geq 0$, then we may effectively use the Thiele algorithm to evaluate $f(\infty)$ via the use of a few values of $f(x)$, for finite x. Indeed, this has been done recently in an ultrasonic tomography algorithm [19].

3.2. Evaluation of the Rationals of the previous Section via the Thiele Algorithm.

Let $\mathbb{P}_n$ denote the family of polynomials of degree $\leq n$, and consider the evaluation of the rational function

$$(3.7) \qquad r(x) = \frac{p_n(x)}{q_{n+\sigma}(x)}$$

for $p_n \in \mathbb{P}_n$, $q_{n+\sigma} \in \mathbb{P}_{n+\sigma}$, where $\sigma = 0$ or 1, and such that

$$(3.8) \qquad \begin{cases} r(x_{2k}) = f(x_{2k}), & k = 0,1,\ldots,n \\ r(x_{2k-1}) = \infty \quad , & k = 1,2,\ldots,n+\sigma. \end{cases}$$

Then

$$(3.9) \qquad \rho(x) = \frac{1}{r(x)}$$

can be evaluated via the Thiele algorithm, using the $2n + 1 + \sigma$ values $\rho(x_k) = 1/r(x_k)$, $k = 0,1,\ldots,2n + 1 + \sigma$. Eq. (3.9) then yields $r(x) = 1/\rho(x)$. In general, there seems to be no guarantee that the ρ algorithm will always work; however, interlacing the zero and non-zero values of ρ in the above fashion has worked, in our experience. Since all of the poles of $r(x)$ have been pre-determined, there are no unwanted poles.

Let us next consider the evaluation of the rational function in (2.10) for the case when $\sigma = 0$, i.e.,

$$r(x) = \Sigma_{j=-N}^{N} \frac{(1-x^2)B(x)f(\zeta_j)}{(x-\zeta_j)(1-\zeta_j^2)B'(\zeta_j)} \; ; \quad B(x) = \Pi_{-N}^{N} \frac{x-\zeta_j}{1-\zeta_j x}$$

(3.10)

$$\zeta_j = \frac{q^j-1}{q^j+1}.$$

Since $\zeta_0 = 0$, $B(x) = p_{2N+1}(x)/q_{2N}(x)$, where $p_{2N+1} \in \mathbf{P}_{2N+1}$, $q_{2N} \in \mathbf{P}_{2N}$. Hence $r(x) = p_{2N+2}(x)/q_{2N}(x)$ and it has $(1-x^2)$ as a factor. Hence

$$(3.11) \qquad \rho(x) = \frac{1-x^2}{r(x)} = \frac{p_{2N}(x)}{q_{2N}(x)}, \qquad p_{2N}, q_{2N} \in \mathbf{P}_{2N} \, ,$$

is completely determined by the 4N + 1 values

$$(x_{2k}, \rho(x_{2k})) = (\zeta_{-N+k}, \frac{1-\zeta_{-N+k}^2}{f(\zeta_{-N+k})}), \quad k = 0,1,\ldots,2N$$

$$(x_{2k-1}, \rho(x_{2k-1})) = \begin{cases} (1/\zeta_{-N+k-1}, 0), & k = 1,2,\ldots,N \\ (1/\zeta_{-N+k}, 0), & k = N+1,\ldots,2N \end{cases}$$

and may thus be evaluated via the Thiele algorithm, as above. The rational function r(x) may then be computed via (3.11), i.e.

$$r(x) = \frac{1-x^2}{\rho(x)}.$$

3.3. The ϵ-Algorithm and the Padé Approximation.

The ϵ-algorithm [15], [23] is described as follows.

Given a sequence of m + 1 numbers S_j $(j = 0,1,\ldots,m)$, define ϵ_i^j by

$$(3.12)\quad \begin{cases} \epsilon_0^j = S_j \quad , & j = 0,1,\ldots,m \\ \epsilon_1^j = \dfrac{1}{\epsilon_0^{j+1} - \epsilon_0^j} \quad , & j = 0,1,\ldots,m-1 \\ \epsilon_i^j = \dfrac{1}{\epsilon_{i-1}^{j+1} - \epsilon_{i-1}^j} + \epsilon_{i-1}^{j+1} & j = 0,1,\ldots,m-i \\ & i = 2,3,\ldots,m. \end{cases}$$

The numbers ϵ_i^j may be used to either predict the limiting value of a function, or to evaluate Padé approximations [24].

For example, if

$$(3.13)\qquad S(x) = L + \Sigma_{k=0}^{\mu} e^{\alpha_k x} \Sigma_{\ell=0}^{\mu_k - 1} d_{k\ell} x^{\ell},$$

if

$$S_j = S(jk),$$

and if

$$(3.14)\qquad M = \Sigma_{k=0}^{\mu} \mu_k$$

then

$$(3.15)\qquad \epsilon_M^0 = L.$$

On the other hand, if S_j is defined by

$$(3.16) \qquad S_j = \Sigma_{i=0}^{j} c_i \eta^i$$

then ϵ_{2k}^{n} yields a Padé approximation [23] i.e.,

$$(3.17) \qquad \epsilon_{2k}^{n} = \frac{p_{m+k-1}(\eta)}{q_k(\eta)}.$$

The results of Sec. 2 of this paper together with the representations (3.13) or (3.16) tell us when we may expect the approximations (3.15) or (3.17) to be accurate, when applied to a function f.

For example, if $f(x) - L$ ($L = f(\infty)$) satisfies any of the conditions in sections 2.4 or 2.5, then by taking $S_j = f(jk)$, we see from (3.13) and (3.15) that ϵ_m^0 will converge rapidly to $L = f(\infty)$.

The representation (3.16) may be considered to be an interpolation of an analytic function $f(z)$, where

$$(3.18) \qquad f(jk) = S_j = \Sigma_{k=0}^{j} c_k x^k.$$

This representation then shows that $f(z) - L$ ($L = f(\infty)$) may be assumed to satisfy any of the conditions of Secs. 2.4 or 2.5, provided, e.g., that

$$(3.19) \qquad |c_k x^k| = O(e^{-\alpha k}), \qquad \alpha > 0.$$

Hence in the case when (3.19) is satisfied, we may expect (3.17) to converge rapidly to $f(\infty) = \Sigma_{k=0}^{\infty} c_k x^k$ as $n \to \infty$.

4. A LOWER BOUND ON THE ERROR OF BEST APPROXIMATION.

Let U denote the unit disc in the complex plane, i.e. $\mathfrak{D}^1_{\pi/2}$ in the notation of Eq. (2.1). That is

$$(4.1) \qquad U = \{\zeta \in \mathbb{C}: |\zeta| < 1\}.$$

Let f satisfy the conditions of Thm. 2.1a. Then F defined by (2.6) is in $H_p(U)$, that is, F is analytic in U, and

$$(4.2) \qquad \|F\|_p = \lim_{r\to 1^-} \left(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^p d\theta\right)^{1/p} < \infty.$$

Let S denote the family of all such functions F such that $\|F\|_p \leq 1$. Then, by Anderssen [1] we have

$$(4.3) \qquad C_1 \exp\{-\pi(\frac{2N}{p'})^{1/2}\} \leq \inf_{p_{2N}, q_{2N+1}} \sup_{F\in S} \left|\int_{-1}^{1} [F(\xi) - \frac{p_{2N}(\xi)}{q_{2N+1}(\xi)}] d\xi\right|$$

where $p' = p/(p-1)$ and where C_1 is a positive constant depending only on p, and p_{2N} and q_{2N+1} denote polynomials of degree $2N$ and $2N + 1$ respectively.

Now if $\eta(\xi)$ is defined by (2.10) then the terms in square brackets on the right hand side of (4.3) is just $\eta(\xi)/(1-\xi^2)$. Hence

$$(4.4) \qquad C_1 \exp\{-\pi(\frac{2N}{p'})^{1/2}\} \leq \left|\int_{-1}^{1} \frac{\eta(\xi)}{1-\xi^2} d\xi\right|.$$

We now split the integral on the right hand side into an integral over $(-\delta,\delta)$, $\delta > 0$ plus an integral over $[-1,1] - (-\delta,\delta)$.

Then

$$(4.5)\qquad \left|\int_{-\delta}^{\delta} \frac{\eta(\xi)}{1-\xi^2}\, d\xi\right| \le \max_{\xi\in[-1,1]} |\eta(\xi)| \int_{-\delta}^{\delta} (1-x^2)^{-1} dx$$

$$= \|\eta\|_\infty \cdot 2 \log\left(\frac{1+\delta}{1-\delta}\right)$$

while from (2.19) we have

$$(4.6)\qquad \left|\int_{[-1,1]-(-\delta,\delta)} \frac{\eta(\xi)}{1-\xi^2}\, d\xi\right| \le C_2 \int_{\delta}^{1} (1-\xi^2)^{-1/p}\, d\xi$$

$$\le C_2 \int_{\delta}^{1} (1-\xi)^{-1/p} d\xi$$

$$= C_2 p'(1-\delta)^{p'}$$

where C_2 is a constant depending on on p.

Hence, by (4.4), (4.5) and (4.6), we get

$$(4.7)\qquad \|\eta\|_\infty \ge \frac{1}{2\log\frac{1+\delta}{1-\delta}} \exp\{-\pi(\tfrac{2N}{p'})^{1/2}\} - 2p'(1-\delta)^{1/p}.$$

Taking $1-\delta = e^{-N}$, and combining with Thm. 2.1a we find that there exists an $N_0 \ge 0$ such that if $N > N_0$, then there are constants C_3 and C_4 such that

$$C_3 N^{-1} \exp\{-\pi(\frac{2N}{p'})^{1/2}\}$$

$$(4.8) \qquad \leq \inf_{p_{2N+2},q_{2N+1}} \sup_{F\in S} \sup_{-1<\xi<1} \left| f(\xi) - \frac{p_{2N+2}(\xi)}{q_{2N+1}(\xi)} \right|$$

$$\leq C_4 N^{1/(2p')} \exp\{-\pi(\frac{N}{2p'})^{1/2}\} .$$

These inequalities show that while the exact lower bound on the left hand side of (4.8) is not known, the results of this paper are in the right "ballpark" with respect to their accuracy.

We therefore conclude with a problem: Given f analytic in U, $F \in H_p(U)$, where $F(z) = f(z)/(1-z^2)$, and given N, can a rational approximation p_N/q_N of f on $[-1,1]$ which is linear in F be as accurate as the best rational approximation to f of the form p_N/q_N? Here p_N and q_N are polynomials of degree at most N.

Appendix A. BLASCHKE PRODUCT ESTIMATES.

We shall see in what follows that the four rational approximation results referred to at the outset of Sec. 1 are simply related via elementary conformal transformations. For example, once we have a rational function approximation in w over $[-1,1]$ we can readily obtain one in z over $(0,\infty)$ via the transformation

$$(A.1) \qquad w = \frac{z-1}{z+1} \quad (\Leftrightarrow z = \frac{1+w}{1-w})$$

which is a conformal map of the right half plane onto the unit disc, and which transforms $p_{n+1}(w)/q_n(w)$ into a rational function $P_{n+1}(z)/Q_{n+1}(z)$. Similarly, starting with a rational function of z for approximation over $[0,\infty]$ we can use the second equation in (A.1) to get a rational approximation in w over $[-1,1]$.

It is most convenient from the point of view of using known results for Jacobi theta functions to first consider approximation over $[0,\infty]$. For purposes of interpolation at the points q^j, $j = -N,\ldots,N$, $0 < q < 1$ it is natural to start with the product

$$\Phi_N^*(z) = \Pi_{j=-N}^{N} \frac{z-q^j}{z+q^j}, \qquad 0 < z < \infty.$$

Unfortunately this product does not have a limit as $N \to \infty$, since the product changes sign with N as N increases. However the alternate form

$$\text{(A.2)} \qquad \Phi_N(z) = \frac{z-1}{z+1} \Pi_{j=1}^{N} \frac{1 - q^j(z+1/z) + q^{2j}}{1 + q^j(z+1/z) + q^{2j}}$$

has the same zeros and poles as Φ_N^*, and moreover, the product converges as $N \to \infty$. Let us set

$$\text{(A.3)} \qquad \Phi(z,q) = \frac{z-1}{z+1} \Pi_{j=1}^{\infty} \frac{1 - q^j(z+1/z) + q^{2j}}{1 + q^j(z+1/z) + q^{2j}}$$

and let us define $m(q)$ by

$$\text{(A.4)} \qquad m(q) = \sup_{0<z<\infty} |\Phi(z,q)|.$$

Next, let us use the standard notation for elliptic functions, for $0 < k < 1$,

$$\text{(A.5)} \quad \begin{cases} u = u(k) = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \Leftrightarrow w = \mathrm{sn}(u;k) \\ \mathrm{cn}(u;k) = \sqrt{1-\mathrm{sn}^2(u;k)}, \quad -K < u < K \\ K = K(k) = u(1) \\ k_1 = \sqrt{1-k^2} \\ K' = K(k_1) \\ q = \exp[-\pi K'/K] \\ q_1 = \exp[-\pi K/K']. \end{cases}$$

We then prove

<u>Lemma A.1</u>: *If $m(q)$ is defined by (A.5), then*

$$\text{(A.6)} \qquad m(q^2) = k_1^{1/2}.$$

<u>Proof</u>: Let us introduce the Jacobi theta functions [9]

$$\text{(A.7)}\quad \left.\begin{aligned} \vartheta_s(u) &= \left(\frac{16\,q}{k^2k_1^2}\right)^{1/6}\sin v\ \Pi_{j=1}^{\infty}(1-2q^{2j}\cos 2v+q^{4j}) \\ \vartheta_c(u) &= \left(\frac{16q\ k_1}{k^2}\right)^{1/6}\cos v\ \Pi_{j=1}^{\infty}(1+2q^{2j}\cos 2v+q^{4j}) \end{aligned}\right\}\ u = \frac{2K}{\pi}v.$$

Then, by (A.3) and (A.7) we have

$$\text{(A.8)}\qquad \Phi(e^{2iv},q^2) = ik_1^{1/2}\,\frac{\vartheta_s(u)}{\vartheta_c(u)}.$$

Now, using Eqs. (16.36.3) and (16.3.3) of [9] we have

$$\text{(A.9)}\qquad \Phi(e^{2iv},q^2) = ik_1^{1/2}\,\frac{\mathrm{sn}(u;k)}{\mathrm{cn}(u;k)}.$$

However, we are interested in bounding $\Phi(z,q)$ on the interval $0 \le z \le \infty$. This means that we want to find the maximum value of $|\Phi(e^{2i(-iv)},q)|$, for $-\infty \le v \le \infty$. However, by Eqs. (16.20.1) and (16.20.2) of [9] we have

$$\text{(A.10)}\qquad \frac{\mathrm{sn}(-iu;k)}{\mathrm{cn}(-iu;k)} = -i\,\mathrm{sn}(u;k_1).$$

But for u real

$$\text{(A.11)}\qquad \max_{-\infty\le u\le\infty} |\mathrm{sn}(u;k_1)| = 1,$$

so that

(A.12) $$\max_{-\infty\leq v\leq\infty} |\Phi(e^{2v},q^2)| = k_1^{1/2}.$$

This completes the proof of Lemma 2.1.

We also introduce the rational function

(A.13) $$\Psi_N(z) = \Pi_{j=1}^{N} \frac{1-q^{j-1/2}(z+1/z) + q^{2j-1}}{1+q^{j-1/2}(z+1/z) + q^{2j-1}}$$

for 2N - point rational approximation at the points $q^{\pm(j-1/2)}$, $j = 1,2,\ldots,N$.

Letting $N \to \infty$, we set

(A.14) $$\Psi(z,q) = \Pi_{j=1}^{\infty} \frac{1 - q^{j-1/2}(z+1/z) + q^{2j-1}}{1 + q^{j-1/2}(z+1/z) + q^{2j-1}},$$

and

(A.15) $$n(q) = \sup_{0<z<\infty} |\Psi(z,q)|.$$

We then prove

Lemma A.2: If $n(q)$ is defined by (A.15), then

(A.16) $$n(q^2) = k_1^{1/2}.$$

Proof: In this case we use the theta functions

(A.17)
$$\theta_d(u) = \left(\frac{k^2k_1^2}{16\,q}\right)^{1/12} \Pi_{j=1}^{\infty}(1 + 2q^{2j-1}\cos 2v + q^{2j-2})$$

$$\theta_n(u) = \left(\frac{k^2}{16qk_1^4}\right)^{1/12} \Pi_{j=1}^{\infty}(1 - 2q^{2j-1}\cos 2v + q^{2j-2})$$

and proceed similarly as the proof of Lemma 2.1, using equations in [9].

Lemma A.3: Let k_1 and q be defined as in Eqs. (A.5). Then

$$(A.18) \qquad k_1^{1/2} \leq 2 \exp\{-\frac{\pi^2}{4 \log \frac{1}{q}}\}.$$

Proof: The following series expansion taken from [8, p. 378] converges for $0 \leq k_1 \leq 1$:

$$(A.19) \qquad q^{1/4} = (\frac{k}{4})^{1/2}[1 + 2(\frac{k}{4})^2 + 15(\frac{k}{4})^4 + 150(\frac{k}{4})^6 + \ldots].$$

Since all of the terms of the series in square brackets are positive, (A.19) taken together with (A.5) implies that

$$(A.20) \qquad k_1^{1/2} \leq 2\, q_1^{1/2}.$$

Since, however, the last two equations in (A.5) yield

$$(A.21) \qquad (\log \frac{1}{q})(\log \frac{1}{q_1}) = \pi^2;$$

we find, after substituting this result into (A.20) and using the fact that $0 < q < 1$, $0 < q_1 < 1$, we obtain (A.18).

Lemma A.4: The functions Φ of (A.4) and Ψ of (A.14) satisfy

$$(A.22) \qquad \sup_{0<z<\infty} |\Phi(z,q)| \leq 2 \exp\{-\frac{\pi^2}{2 \log \frac{1}{q}}\}$$

$$\text{(A.23)} \qquad \sup_{0<z<\infty} |\Psi(z,q)| \le 2 \exp\{-\frac{\pi^2}{2 \log \frac{1}{q}}\}.$$

Proof: As a consequence of Lemmas A.1, A.2 and A.3, both functions $\Phi(z,q^2)$ and $\Psi(z,q^2)$ are bounded on $[0,\infty]$ by $2 \exp[-\pi^2/(4 \log \frac{1}{q})]$. Hence, replacing q^2 by q we get (A.22) and (A.23).

We remark that while we used elliptic functions to derive (A.22) and (A.23) these inequalities do not involve elliptic functions.

Lemma A.5: *Let* σ *be either* 0 *or* 1/2. (a) *If* $|z| \ge \xi \ge q^{N+1/2-\sigma}$, *and if*

$$\text{(A.24)} \qquad t_N \equiv \Pi_{j=N+1}^{\infty} \frac{z+q^{j-\sigma}}{z-q^{j-\sigma}}$$

then

$$\text{(A.25)} \qquad |t_N| \le \exp\{\frac{\pi^2}{4 \log \frac{1}{q}} \frac{q^{N+1/2-\sigma}}{\xi}\}.$$

(b) *If* $0 < |z| \le \zeta \le q^{-N-1/2+\sigma}$, *and if*

$$\text{(A.26)} \qquad \tau_M \equiv \Pi_{j=M+1}^{\infty} \frac{z^{-1} + q^{j-\sigma}}{z^{-1} - q^{j-\sigma}}$$

then

$$\text{(A.27)} \qquad |\tau_M| \le \exp\{\frac{\pi^2}{4 \log \frac{1}{q}} \zeta q^{M+1/2-\sigma}\}.$$

Proof: The proof of (A.27) is almost identical to the proof of (A.25), and hence we shall only prove (A.25). We have

$$\log|t_N| = \mathrm{Re}\,\Sigma_{n=0}^{\infty} \frac{2}{2n+1} \Sigma_{j=N+1}^{\infty} \left(\frac{q^{j-\sigma}}{z}\right)^{2n+1}$$

(A.28)
$$\leq \Sigma_{n=0}^{\infty} \frac{2}{2n+1} \Sigma_{j=N+1}^{\infty} \left(\frac{q^{j-\sigma}}{|z|}\right)^{2n+1}$$

$$\leq \Sigma_{n=0}^{\infty} \frac{2}{2n+1} \Sigma_{j=N+1}^{\infty} \left(\frac{q^{j-\sigma}}{\xi}\right)^{2n+1}$$

$$= \Sigma_{n=0}^{\infty} \frac{2}{2n+1} \left(\frac{N+1/2-\sigma}{\xi}\right)^{2n+1} \frac{q^{n+1/2}}{1-q^{2n+1}}.$$

But

(A.29) $$\frac{q^{n+1/2}}{1-q^{2n+1}} = \frac{1}{2\sinh[(n+\frac{1}{2})\log\frac{1}{q}]} \leq \frac{1}{(2n+1)\log\frac{1}{q}}.$$

Hence

(A.30) $$\log|t_N| \leq \frac{q^{n+1/2-\sigma}}{\xi\log\frac{1}{q}} \Sigma_{n=0}^{\infty} \frac{2}{(2n+1)^2} = \frac{q^{N+1/2-\sigma}}{\xi\log\frac{1}{q}} \frac{\pi^2}{4}.$$

This is just the logarithmic form of (A.25).

Lemma A.6: Let $|z| > 0$, $-\pi < \theta < \pi$, and let $z = |z|e^{i\theta}$. Then

(A.31) $$\int_0^{\infty} \frac{1}{t} \log\left|\frac{z+t}{z-t}\right| dt = \pi\left(\frac{\pi}{2} - |\theta|\right).$$

Proof: Splitting the integral as an integral from 0 to z plus an integral from $|z|$ to ∞ we have

$$\text{(A.32)} \qquad \mathrm{Re}\int_0^{|z|}\frac{1}{t}\log\frac{z+t}{z-t}\,dt = \mathrm{Re}\int_0^{|z|}\frac{1}{t}\Sigma_{n=0}^{\infty}\frac{2}{2n+1}\left(\frac{t}{z}\right)^{2n+1}dt$$

$$= \mathrm{Re}\,\Sigma_{n=0}^{\infty}\frac{2}{(2n+1)^2}e^{-(2n+1)i\theta} = \Sigma_{n=0}^{\infty}\frac{2}{(2n+1)^2}\cos(2n+1)\theta.$$

Similarly,

$$\text{(A.33)} \qquad \mathrm{Re}\int_{|z|}^{\infty}\frac{1}{t}\log\frac{t+z}{t-z}\,dt = \Sigma_{n=0}^{\infty}\frac{2}{(2n+1)^2}\cos(2n+1)\theta.$$

Hence

$$\text{(A.34)} \qquad \int_0^{\infty}\frac{1}{t}\log\left|\frac{z+t}{z-t}\right|dt = 2\,\Sigma_{n=0}^{\infty}\frac{2}{(2n+1)^2}\cos(2n+1)\theta$$

$$= \pi\left(\frac{\pi}{2} - |\theta|\right)$$

from Fourier series.

<u>Corollary A.7</u>: *If* z *satisfies the conditions of Lemma A.6 and if* $\mathcal{P} = \{t: t = \rho e^{i\varphi},\ \varphi$ *fixed*, $0 \le \rho \le \infty\}$, *then*

$$\text{(A.35)} \qquad \int_{\mathcal{P}}\left|\frac{1}{t}\log\frac{z+t}{z-t}\,dt\right| \le \frac{\pi^2}{2}.$$

<u>Proof:</u> The proof is similar to the proof of Lemma A.5, by proceeding along the lines of equations (A.31) to (A.34), dropping the real parts, and replacing t by ρ and z by $|z|$.

<u>Lemma A.8</u>: *Let* $0 < |\theta| < \pi$, *let* $z = |z|e^{\pm i\theta}$, *and define* $\epsilon(\theta,q)$ *by*

$$\text{(A.36)} \qquad \varepsilon(\theta,q) = \frac{\pi^2}{2}\left[\frac{\exp\left(-\frac{2\pi|\theta|}{\log\frac{1}{q}}\right)}{1-\exp\left(-\frac{2\pi|\theta|}{\log\frac{1}{q}}\right)} + \frac{\exp\left(-\frac{2\pi(\pi-|\theta|)}{\log\frac{1}{q}}\right)}{1-\exp\left(-\frac{2\pi(\pi-|\theta|)}{\log\frac{1}{q}}\right)}\right]$$

<u>If</u> σ <u>is</u> <u>either</u> 0 <u>or</u> 1/2, <u>then</u>

$$\text{(A.37)} \qquad \left|\pi\left(\frac{\pi}{2} - |\theta|\right) - \log\frac{1}{q}\,\Sigma_{j=-\infty}^{\infty}\log\left|\frac{z+q^{j+\sigma}}{z-q^{j+\sigma}}\right|\right| \le \varepsilon(\theta,q).$$

<u>Proof</u>: Without loss of generality, let us fix θ in the range $0 < \theta < \pi$. The function

$$\text{(A.38)} \qquad f(t) = \frac{1}{t}\log\frac{z + tq^{\sigma}}{z - tq^{\sigma}}$$

is then an analytic function of t in the sector $\theta - \pi < \arg t < \theta$, it is absolutely integrable along any ray $t = |t|e^{i\varphi}$, $0 \le |t| \le \infty$, $\theta - \pi \le \varphi \le \theta$, and moreover for t in this sector of analyticity. $f(t) = O(1)$ as $t \to 0$, $f(t) = O(t^{-2})$ as $t \to \infty$. Hence if we can apply Thm. 4.3 of [17], Lemma A.6 and Corollary A.7 to get the result (A.36) for $0 < \theta < \pi$. The proof for the case when $-\pi < \theta < 0$ is similar.

<u>Corollary A.9</u>: <u>If</u> $z = |z|e^{i\theta}$, <u>where</u> $0 < |\theta| < \pi$, <u>if</u> $\varepsilon(\theta,q)$ <u>is</u> <u>defined</u> <u>by</u> (A.36) <u>and</u> <u>if</u> σ <u>is</u> <u>either</u> 0 <u>or</u> 1/2, <u>then</u>

$$\text{(A.39)} \qquad \exp\left\{\frac{\pi(\frac{\pi}{2}-|\theta|)-\varepsilon(\theta,q)}{\log\frac{1}{q}}\right\} \le \Pi_{j=-\infty}^{\infty}\left|\frac{z+q^{j+\sigma}}{z-q^{j+\sigma}}\right|$$

$$\le \exp\left\{\frac{\pi(\frac{\pi}{2}-|\theta|)+\varepsilon(\theta,q)}{\log\frac{1}{q}}\right\}.$$

<u>Theorem A.10</u>: <u>Let</u> $z = |z|e^{i\theta}$, <u>let</u> $\varepsilon(\theta,q)$ <u>be defined by</u> (A.36) <u>and let</u> σ <u>be either</u> 0 <u>or</u> 1/2.

(i) <u>If</u> $0 < |\theta| \le \frac{\pi}{2}$, <u>then</u>*

$$\text{(A.40)} \qquad \Pi_{j=-M+2\sigma}^{N}\left|\frac{z+q^{j-\sigma}}{z-q^{j-\sigma}}\right| \le \exp\left\{\frac{\pi(\frac{\pi}{2}-|\theta|)+\varepsilon(\theta,q)}{\log\frac{1}{q}}\right\}$$

(ii) <u>If</u> $\pi/2 < |\theta| < \pi$, <u>let</u> ξ <u>and</u> ζ <u>be selected such that</u>

$$\text{(A.41)} \qquad q^{N+1/2-\sigma} \le \xi < \zeta \le q^{-(M+1/2-\sigma)}.$$

<u>If</u> $|z|$ <u>lies in the range</u> $\xi \le |z| \le \zeta$, <u>then</u>

$$\text{(A.42)} \qquad \Pi_{j=-M+2\sigma}^{N}\left|\frac{z+q^{j-\sigma}}{z-q^{j-\sigma}}\right| \le \exp\left\{\frac{\pi(\frac{\pi}{2}-|\theta|)+\varepsilon(\theta,q)+\frac{\pi^2}{4}(q^{N+\frac{1}{2}-\sigma}/\xi+q^{M+\frac{1}{2}-\sigma}\zeta)}{\log\frac{1}{q}}\right\}.$$

(iii) <u>If</u> $\theta = \pi$, <u>and</u> z <u>lies in the range</u> $\xi \le |z| \le \zeta$, <u>where</u> ξ <u>and</u> ζ <u>satisfy</u> (2.41), <u>then</u>

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*If $|\theta| = \frac{\pi}{2}$, the left-hand side of (A.40) is identically 1.

(A.43) $$\Pi_{j=M+2\sigma}^{N}\left|\frac{z+q^{j-\sigma}}{z-q^{j-\sigma}}\right| \leq 2\exp\{\frac{-\frac{\pi^2}{2}+\frac{\pi^2}{4}(q^{N+\frac{1}{2}-\sigma}/\xi + q^{M+\frac{1}{2}-\sigma}\zeta)}{\log\frac{1}{q}}\};$$

<u>or equivalently</u>, <u>if</u> $\xi \leq z \leq \zeta$, <u>then</u>

(A.44) $$\Pi_{j=-M+2\sigma}^{N}\left|\frac{z-q^{j-\sigma}}{z+q^{j-\sigma}}\right| \leq 2\exp\{\frac{-\frac{\pi^2}{2}+\frac{\pi^2}{4}(q^{N+\frac{1}{2}-\sigma}/\xi + q^{M+\frac{1}{2}-\sigma}\zeta)}{\log\frac{1}{q}}\}.$$

<u>Proof</u>: (i) We have

(A.45) $$\Pi_{j=-M+2\sigma}^{N}\left|\frac{z+q^{j-\sigma}}{z-q^{j-\sigma}}\right| = \Pi_{j=-\infty}^{\infty}\left|\frac{z+q^{j-\sigma}}{z-q^{j-\sigma}}\right|\Pi_{j<-M+2\sigma,\, j>N}\left|\frac{z-q^{j-\sigma}}{z+q^{j-\sigma}}\right|.$$

However since $|\theta| = |\arg z| \leq \pi/2$ each term $(z-q^{j-\sigma})/(z+q^{j-\sigma})$ of the product on the right hand side of (A.45) is at most 1 in magnitude. Hence (A.40) then follows from (A.39).

(ii) This is also a consequence of (A.45), (A.39), and Lemma A.5.

(iii) Both $|\Phi(z,q)|$ (by taking $\sigma = 0$) and $|\Psi(z,q)|$ (by taking $\sigma = 1/2$) are expressible by means of the infinite product $\Pi_{j=-\infty}^{\infty}|(z+q^{j-\sigma})/(z-q^{j-\sigma})|$. Hence the inequalities (A.43) and (A.44) follow from Lemmas A.4 and A.5.

Appendix B: CONTOUR INTEGRAL ESTIMATES.

The hypergeometric function

$$\text{(B.1)} \qquad F(a,b;c;z) = \Sigma_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \; n!} z^n, \qquad |z| < 1$$

may also be expressed via the following integrals provided that $c > b > 0$

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z)$$

$$\text{(B.2)} \qquad = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$$

$$\text{(B.3)} \qquad = 2\int_0^{\pi/2} \sin^{2b-1}\theta \cos^{2c-2b-1}\theta (1-z\sin^2\theta)^{-a} d\theta$$

$$\text{(B.4)} \qquad = 2\int_0^{\pi/2} \cos^{2b-1}\theta \sin^{2c-2b-1}\theta (1-z\cos^2\theta)^{-a} d\theta$$

$$\text{(B.5)} \qquad = \int_0^{\infty} u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a} du.$$

If also $c - b - a > 0$, then

$$\text{(B.6)} \qquad F(a.b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Other identities involving the function $F(a.b;c;z)$ which we shall use are

$$\text{(B.7)} \qquad F\left(a,b;2b;\frac{4z}{(1+z)^2}\right) = (1+z)^{2a} F\left(a,a-b+\tfrac{1}{2};b+\tfrac{1}{2};z^2\right)$$

[8, p. 50]

(B.8) $$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a;c-b;c;z)$$

[8, p. 47]

(B.9) $$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)$$

$$+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z)$$

[8, p. 47]

(B.10) $$F(a,b;c;z) = (1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F(a,c-b;a-b+1;\frac{1}{1-z})$$

$$+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F(b,c-a;b-a+1;\frac{1}{1-z})$$

[8, p. 48]

(B.11) $$F(a,b;c;z) = (1-z)^{-a}F(a,c-b;c;\frac{-z}{1-z})$$

$$= (1-z)^{-b}F\left(b,c-a;c;-\frac{z}{1-z}\right)$$

[8, p. 47].

Some gamma function identies will also be convenient:

(B.13) $$\Gamma(z)\Gamma(z+\frac{1}{2}) = \pi^{1/2}2^{1-2z}\Gamma(2z)$$

(B.14) $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

(B.15) $$\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos(\pi z)}.$$

Let Σ_d^1 be defined by Eq. (2.1) (See Fig. 2.1).

The Integrals to be Estimated

We shall estimate the integral

$$(B.16) \qquad G(p',d,x) \equiv \frac{1}{2\pi}\int_{\partial\mathcal{D}_d^1} |\zeta-x|^{-p'}|d\zeta| \qquad \begin{array}{l} 1 < p' < \infty, \\ -1 < x < 1, \\ 0 < d \le \pi/2 \end{array}$$

for $1 < p' < \infty$, $-1 < x < 1$, and $0 < d \le \pi/2$, and we shall estimate the integral

$$(B.17) \qquad H(\alpha,\beta,d,x) \equiv \frac{1}{2\pi}\int_{\partial\mathcal{D}_d^1} |1+\zeta|^{\alpha-1}|1-\zeta|^{\beta-1}|\zeta-x|^{-1}|d\zeta|$$

for $0 < \alpha < 1$, $0 < \beta < 1$, $-1 < x < 1$, and $0 < d \le \pi/2$.

<u>Theorem B.1</u>: <u>Let</u> $G(p',d,x)$ <u>be defined by</u> (B.16). <u>Then there exists a constant</u> C <u>depending only on</u> p' <u>and</u> d <u>such that</u>

(B.18) $\quad G(p',d,x) \leq C(1-x^2)^{1-p'}, \quad 0 \leq x < 1.$

<u>Proof</u>: Let us set

(B.19) $\quad G(x) \equiv G(p',\frac{\pi}{2},x).$

Then

$$G(x) = \frac{1}{2\pi}\int_0^{2\pi}[1-2x\cos\theta + x^2]^{-p'/2}\,d\theta \tag{B.20}$$

$$= \frac{2}{\pi}(1+x)^{-p'}\int_0^{\pi/2}[1-\frac{4x}{(1+x)^2}\cos^2\theta]^{-p'/2}\,d\theta$$

$$= (1+x)^{-p'}F(\frac{p'}{2},\frac{1}{2};1;\frac{4x}{(1+x)^2}) \qquad \text{(by (B.4))}$$

$$= F(\frac{p'}{2},\frac{p'}{2};1;x^2) \qquad \text{(by (B.7))}$$

$$= (1-x^2)^{1-p'}F(1-\frac{p'}{2},1-\frac{p'}{2};1;x^2) \qquad \text{(by (B.8))} \tag{B.21}$$

$$\leq (1-x^2)^{1-p'}\frac{\Gamma(p'-1)}{\Gamma(\frac{p'}{2})\Gamma(\frac{p'}{2})} \tag{B.22}$$

by (B.6) and the positive coefficient expansion (B.1) of $F(1-p'/2,1-p'/2;1;x^2)$.

This proves (B.18) for the case when $d = \pi/2$.

We next consider the case of $0 < d < \pi/2$.

<u>The case $0 < d < \pi/2$.</u>

The region $\mathfrak{D}_d^{1}$ is now the intersection of the two discs (2.2). The contour integral (B.16) is thus an integral over parts of two circles. If we replace the integrals over these parts by the integrals over the whole circles, we find that

$$\text{(B.23)} \qquad G(p',d,x) \leq \frac{2}{2\pi}\int_{|\zeta|=\csc d} |\zeta-\sqrt{x^2+\cot^2 d}|^{-p'}|d\zeta|$$

$$\text{(B.24)} \qquad = 2(\csc d)^{1-p'}\frac{1}{2\pi}\int_{|\zeta|=1} |\zeta-\sin d\sqrt{x^2+\cot^2 d}|^{-p'}|d\zeta|$$

$$\text{(B.25)} \qquad = 2(\csc d)^{1-p'}\, G\left(\sin d\sqrt{x^2+\cot^2 d}\right).$$

(Compare (B.16), (B.)). The . inequality (B.18) now follows by proceeding as in (B.20) - (B.22).

Let $H(\alpha,\beta,d,x)$ be defined by (B.17), and let us set

$$\text{(B.26)} \qquad H(x) = \frac{1}{2\pi}\int_{|\zeta|=1} |1+\zeta|^{\alpha-1}|1-\zeta|^{\beta-1}|\zeta-x|^{-1}|d\zeta|.$$

Then, setting $\zeta = e^{2i\theta}$, $0 \leq \theta \leq \pi$, we get

$$\text{(B.27)} \qquad H(x) = \frac{2^{\alpha+\beta-1}}{\pi(1-x)}\int_0^{\pi/2}\cos^{\alpha-1}\theta\sin^{\beta-1}\theta\left[1+\frac{4x}{(1-x)^2}\sin^2\theta\right]^{-1/2}d\theta$$

$$\text{(B.28)} \qquad = \frac{2^{\alpha+\beta-2}}{\pi(1-x)}\,\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})}F\left(\frac{1}{2},\frac{\beta}{2};\frac{\alpha+\beta}{2};-\frac{4x}{(1-x)^2}\right)$$

by (B.3). Then (B.9) yields

$$\text{(B.29)} \qquad \begin{aligned} H(x) &= \frac{2^{\alpha+\beta-2}}{\pi(1-x)}\,\frac{\Gamma(\frac{\alpha-1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta-1}{2})}F\left(\frac{1}{2},\frac{\beta}{2};\frac{3-\alpha}{2};\left(\frac{1+x}{1-x}\right)^2\right) \\ &+ \frac{2^{\alpha+\beta-2}}{\pi}(1+x)^{\alpha-1}(1-x)^{-\alpha}\,\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{1}{2})}F\left(\frac{\alpha+\beta-1}{2},\frac{\alpha}{2};\frac{\alpha+1}{2};\left(\frac{1+x}{1-x}\right)^2\right). \end{aligned}$$

On the other hand, replacing sin θ by cos φ in (B.27) and then using (B.4), we get

$$H(x) = \frac{2^{\alpha+\beta-2}}{\pi(1+x)} \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta-1}{2})}{\Gamma(\frac{\alpha+\beta-1}{2})} F(\frac{1}{2},\frac{\alpha}{2};\frac{3-\beta}{2};(\frac{1-x}{1+x})^2)$$

(B.30)

$$+ \frac{2^{\alpha+\beta-2}}{\pi}(1-x)^{\beta-1}(1+x)^{-\beta}\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{1-\beta}{2})}{\Gamma(\frac{1}{2})} F(\frac{\alpha+\beta-1}{2},\frac{\beta}{2};\frac{\beta+1}{2};(\frac{1-x}{1+x})^2).$$

Let us next transform (B.17) by means of the transformations

(B.31) $$\zeta = \frac{w-1}{w+1}, \qquad x = \frac{u-1}{u+1}$$

to get

(B.32) $$H(x) = \frac{2^{\alpha+\beta-1}(1+u)}{2\pi}\int_0^\infty w^{\alpha-1}(1+w^2)^{\frac{1-\alpha-\beta}{2}}(w^2+u^2)^{-1/2}dw$$

$$= \frac{2^{\alpha+\beta-3}}{\pi}(1+u)\frac{\Gamma(\frac{\alpha-1}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta-1}{2})} F(\frac{1}{2},\frac{\beta}{2};\frac{3-\alpha}{2};u^2)$$

$$+ \frac{2^{\alpha+\beta-3}}{\pi}(1+u)u^{\alpha-1}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1-\alpha}{2})}{\Gamma(\frac{1}{2})} F(\frac{\alpha+\beta-1}{2},\frac{\alpha}{2};\frac{\alpha+1}{2};u^2)$$

from (B.29), and

$$\text{(B.33)}\quad \begin{aligned} H(x) &= \frac{2^{\alpha+\beta-3}}{\pi}\,\frac{1+u}{u}\,\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta-1}{2})}{\Gamma(\frac{\alpha+\beta-1}{2})}F(\tfrac{1}{2},\tfrac{\alpha}{2};\tfrac{3-\beta}{2},u^{-2}) \\ &+ \frac{2^{\alpha+\beta-3}}{\pi}u^{-\beta-1}(1+u)\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{1-\beta}{2})}{\Gamma(\frac{1}{2})}F(\tfrac{\alpha+\beta-1}{2},\tfrac{\beta}{2};\tfrac{\beta+1}{2};u^{-2}), \end{aligned}$$

from (B.30).

The representation (B.32) is convenient on the interval $-1 < x \leq 0$, while the representation (B.33) is convenient on the interval $0 \leq x < 1$.

Now if $-1 < x \leq 0$, $(\frac{1+x}{1-x})^2 \leq 1$; by inspection of the power series of the hypergeometric function in (B.29) and recalling that $0 < \alpha < 1$, $0 < \beta < 1$, we find that each takes on its maximum value if $(1 + x)^2/(1 - x)^2 = 1$. Hence, applying (B.6), we get

$$\text{(B.34)}\quad \begin{aligned} H(x) &\leq \frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{2-\alpha-\beta}{2})}{\Gamma(\frac{\alpha+\beta-1}{2})\Gamma(\frac{2-\alpha}{2})\Gamma(\frac{3-\alpha-\beta}{2})}\,\frac{2^{\alpha-\beta-2}}{\pi} \\ &+ \frac{(1-x)^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{2-\alpha-\beta}{2})2^{\alpha+\beta-2}}{\pi^2\Gamma(\frac{2-\beta}{2})} \end{aligned}$$

if $-1 < x \leq 0$

$$\text{(B.35)} \qquad H(x) \leq \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1+\beta}{2})\Gamma(\frac{1-\beta}{2})\Gamma(\frac{2-\alpha-\beta}{2})}{\Gamma(\frac{\alpha+\beta-1}{2})\Gamma(\frac{2-\alpha}{2})\Gamma(\frac{3-\alpha-\beta}{2})} \frac{2^{\alpha+\beta-2}}{\pi}$$

$$+ \frac{(1-x)^{\beta-1}\Gamma(\frac{\beta}{2})\Gamma(\frac{1-\beta}{2})(\frac{1+\beta}{2})\Gamma(\frac{2-\alpha-\beta}{2})2^{\alpha+\beta-2}}{\pi^2\Gamma(\frac{2-\alpha}{2})}$$

if $0 \leq x < 1$.

Hence, denoting by C the maximum of the right hand side of (B.34) and (B.35) , we have

$$\text{(B.36)} \qquad H(x) \leq C(1+x)^{\alpha-1}(1-x)^{\beta-1}.$$

We now consider the case $0 < d < \pi/2$. In this case, we consider the integral obtained by using (B.31) in (B.17),

$$\text{(B.37)} \qquad H(\alpha,\beta,d,x) = \frac{2^{\alpha+\beta-1}(1+u)}{2\pi}\int_0^{\infty e^{id}} |w|^{\alpha-1}|1+w|^{1-\alpha-\beta}|w-u|^{-1}|dw|.$$

Here, $|w|^{\alpha-1}|1+w|^{1-\alpha-\beta}$ is a decreasing function of $|w|$ along the path of integration, while $|w-u|$ first increases to

$(u \sin d)^{-1}$ and then decreases again. Hence, since $u \sin d$ is the distance from $u + i0$ to the path of integration, we have by rearrangement of the functions, that

$$\text{(B.38)} \qquad H(\alpha,\beta,d,x) \leq \frac{2^{\alpha+\beta}(1+u)}{2\pi} \int_0^\infty w^{\alpha-1}[1+w^2]^{\frac{1-\alpha-\beta}{2}} [w^2+u^2\sin^2 d]^{-\frac{1}{2}} dw$$

$$= 2H(u \sin d)\frac{1 + u}{1+u \sin d}$$

where the last identity was obtained using (B.32).

Theorem B.2: Let $H(\alpha,\beta,d,x)$ be defined by (B.17), where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < d \leq \pi/2$. Then there exists a constant C depending only on α,β and d, such that for $-1 < x < 1$,

$$\text{(B.42)} \qquad H(\alpha,\beta,d,x) \leq C(1+x)^{\alpha-1}(1-x)^{\beta-1}.$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper gives explicit rational functions for interpolating and approximating functions on the intervals $[-1,1]$, $[0,\infty]$ and $[-\infty,\infty]$. The rational functions are linear in the function f to be approximated. The error of approximation of these rationals is nearly as small as the error of best rational

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) approximation with numerator and denominator polynomials of the same degree. Regions of analylicity of f are described, which make it possible to tell a priori, the accuracy which we can expect from rational approximation. A sample result is the following: Let f be analytic in the sector $\mathcal{D} \equiv \{z \in \mathbb{C}: |\arg z| < d \leq \pi/2\}$, define F by $F(z) = (1+z)^2 f(z)/z$. For some p in the range $1 < p < \infty$, let

$$\|F\|_p \equiv \sup_{|\theta|<d} \left[\int_0^\infty |F(te^{i\theta})|^p \frac{dt}{|1+te^{i\theta}|^2}\right]^{1/p} < \infty .$$

Let $p' = p/(p-1)$, and corresponding to a positive integer N, let q be defined by

$$q = \exp\{-\pi[p'/(2N)]^{1/2}\} .$$

Let $B(t)$ be defined by

$$B(t) = \prod_{j=-N}^{N} \frac{t-q^j}{t+q^j} .$$

Then for all $t \in [0,\infty]$,

$$\left| f(t) - \frac{t\,B(t)}{1+t} \sum_{j=-N}^{N} \frac{(1+q^j)f(q^j)}{q^j(t-q^j)B'(q^j)} \right| < C\,N^{1/(2p')} \exp\{-d(\tfrac{2N}{p'})^{1/2}\} \|F\|_p ,$$

where C is a constant depending only on p and d.

LM